

Sign-changing solutions for the stationary Kirchhoff problems involving the fractional Laplacian in \mathbb{R}^N *

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Abstract

In this paper, we study the existence of least energy sign-changing solutions for a Kirchhoff-type problem involving the fractional Laplacian operator. By using the constraint variational method and quantitative deformation lemma, we obtain a least energy nodal solution u_b for the given problem. Moreover, we show that the energy of u_b is strictly larger than twice the ground state energy. We also give a convergence property of u_b as $b \searrow 0$, where b is regarded as a parameter.

Keywords: Kirchhoff equation, Fractional Laplacian, Sign-changing solutions.

1 Introduction and main results

Here we consider the existence of least energy sign-changing solutions for the Kirchhoff type problem involving a fractional Laplacian operator as below:

$$\left(a + b \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy \right) (-\Delta)^s u + V(x)u = f(x, u) \quad \text{in } \mathbb{R}^N, \quad (1.1)$$

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where $s \in (0, 1)$ is fixed, $n > 2s$, a and b are positive constants. $(-\Delta)^s$ is the fractional Laplacian operator which (up to normalization factors) may be defined along a function $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ as

$$(-\Delta)^s \phi(x) = 2 \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{\phi(x) - \phi(y)}{|x - y|^{n+2s}} dy$$

for $x \in \mathbb{R}^N$, where $B_\varepsilon(x) := \{y \in \mathbb{R}^N : |x - y| < \varepsilon\}$. For an elementary introduction to the fractional Laplacian and fractional Sobolev spaces we refer the reader to [16, 27].

Throughout the paper, we make the following assumptions concerning the potential function $V(x)$:

(V₁) $V \in \mathcal{C}(\mathbb{R}^N)$ satisfies $\inf_{x \in \mathbb{R}^N} V(x) \geq V_0 > 0$, where V_0 is a positive constant;

(V₂) There exists $h > 0$ such that $\lim_{|y| \rightarrow \infty} \text{meas}(\{x \in B_h(y) : V(x) \leq c\}) = 0$ for any $c > 0$;

where $B_R(x)$ denotes any open ball of \mathbb{R}^N centered at x with radius $R > 0$, and $\text{meas}(A)$ denotes the Lebesgue measure of set A . The condition (V₂), which is weaker than the coercivity assumption: $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, was first introduced by Bartsch-Wang [6] to overcome the lack of compactness. Moreover, on the nonlinearity f we assume

(f₁) $f : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is a *Carathéodory* function and $f(x, s) = o(|s|^3)$ as $s \rightarrow 0$;

(f₂) For some constant $p \in (4, 2_s^*)$, $\lim_{s \rightarrow \infty} \frac{f(x, s)}{s^{p-1}} = 0$, where $2_s^* = +\infty$ for $N \leq 2s$ and $2_s^* = \frac{2N}{N-2s}$ for $N > 2s$;

(f₃) $\lim_{s \rightarrow \infty} \frac{F(x, s)}{s^4} = +\infty$ with $F(s) = \int_0^s f(t) dt$;

(f₄) $\frac{f(s)}{|s|^3}$ is nondecreasing on $\mathbb{R} \setminus \{0\}$.

The motivation to study problem (1.1) comes from Kirchhoff equations of the type

$$-\left(a + b \int_{\Omega} |\nabla u|^2 dx\right) \Delta u = f(x, u) \quad \text{in } \Omega, \quad (1.2)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $a > 0$, $b \geq 0$ and u satisfies some boundary conditions. The problem (1.2) is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \left(a + b \int_{\Omega} |\nabla u|^2 dx\right) \Delta u = f(x, u), \quad (1.3)$$

which was introduced by Kirchhoff [19] as a generalization of the well-known D'Alembert wave equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L} \int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right) = f(x, u) \quad (1.4)$$

for free vibration of elastic strings. Here, L is the length of the string, h is the area of the cross section, E is the Young modulus of the material, ρ is mass density and p_0 is the initial tension. The problem (1.4) has been proposed and studied as the fundamental equation for understanding several physical systems, where u describes a process which depends on its average. It has also been used to model certain phenomena in biological systems (see [13]).

After the pioneer work of Lions [22], where a functional analysis approach was proposed to Eq.(1.3) with Dirichlet boundary condition, the Kirchhoff equations (1.3) began to receive attention from many researchers. Recently, there are fruitful studies towards problem (1.2), especially on the existence of global classical solution, positive solutions, multiple solutions and ground state solutions, see for example [1, 14, 18, 23] and the references therein. For sign-changing solutions, Zhang et al. [24, 37] used the method of invariant sets of descent flow to obtain the existence of sign-changing solution of (1.2). By using the constraint variation methods and the quantitative deformation lemma, Shuai [31] studied the existence of least energy sign-changing solution for problem (1.2). For the other work about sign-changing solution of Kirchhoff type problem (1.2), we refer the reader to [15, 25] and the reference therein.

On the other hand, we can also start the investigation to problem (1.1) from the direction of nonlinear fractional Schrödinger equation which was proposed by Laskin [20, 21]. In recent years, there are plenty of research on the fractional Schrödinger equation,

$$(-\Delta)^s u + V(x)u = f(x, u) \text{ in } \mathbb{R}^N. \quad (1.5)$$

For the existence, multiplicity and behavior of solutions to Eq.(1.5), we refer to [4, 9–11, 17, 30, 33, 34] and the references therein. In the remarkable work of Caffarelli-Silvestre [11], the authors expressed the nonlocal operator $(-\Delta)^s$ as a Dirichlet-Neumann map for a certain elliptic boundary value problem with local differential operators defined on the upper half space. The technique in [11] is a powerful tool for the study of the equations involving fractional operators. Later on, by using this technique, Chang-Wang [12] obtained a sign-changing solution via the invariant sets of descent flow.

Before presenting our main result, let us first recall the *usual* fractional Sobolev space $H^s(\mathbb{R}^N)$ as follows

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) \mid \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy < \infty \right\},$$

equipped with the inner product

$$\langle u, v \rangle_{H^s(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)}^2 + \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy,$$

and the corresponding norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\|u\|_{L^2(\mathbb{R}^N)}^2 + \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}}.$$

It is well known that $(H^s(\mathbb{R}^N), \|u\|_{H^s(\mathbb{R}^N)})$ is a uniformly convex Hilbert space and the embedding $H^s(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is continuous for any $q \in [2, 2_s^*]$ (See [16]). Moreover, in light of Proposition 3.2 in [16], for $u \in H^s(\mathbb{R}^N)$ we have

$$\|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^N)}^2 = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy. \quad (1.6)$$

In this paper, we denote the fractional Sobolev space for (1.1) by

$$H = \left\{ u \in H^s(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x) u^2(x) dx < +\infty \right\},$$

where the inner product is given by

$$\langle u, v \rangle_H = \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy + \frac{1}{a} \int_{\mathbb{R}^N} V(x) u(x) v(x) dx, \quad (1.7)$$

and the associated norm is

$$\|u\|_H = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + \frac{1}{a} \int_{\mathbb{R}^N} V(x) |u(x)|^2 dx \right)^{\frac{1}{2}}. \quad (1.8)$$

It is readily seen that $(H, \|\cdot\|_H)$ is a uniformly convex Hilbert space and $\mathcal{C}_0^\infty(\mathbb{R}^N) \subset H$ (see [29]). Moreover, thanks to (1.6), the norm $\|\cdot\|_H$ can be also written as

$$\|u\|_H = \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 dx + \frac{1}{a} \int_{\mathbb{R}^N} V(x) u(x)^2 dx \right)^{\frac{1}{2}}. \quad (1.9)$$

Now we give the definition of a *weak* solution to (1.1):

Definition 1.1. We say that $u \in H$ is a *weak* solution of (1.1), if

$$\begin{aligned} \left(a + b \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 dx \right) \int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} \varphi dx \\ + \int_{\mathbb{R}^N} V(x) u(x) \varphi(x) dx = \int_{\mathbb{R}^N} f(x, u) \varphi(x) dx, \end{aligned} \quad (1.10)$$

for all $\varphi \in H$.

And we will omit *weak* throughout this paper for convenience. Define the corresponding functional energy $I_b : H \rightarrow \mathbb{R}$ to problem (1.1) as below:

$$I_b(u) = \frac{a}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 dx + \frac{b}{4} \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 dx \right)^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2 dx - \int_{\mathbb{R}^N} F(x, u) dx. \quad (1.11)$$

It is easy to see that I_b belongs to $\mathcal{C}^1(H, \mathbb{R})$ and the critical points of I_b are the solutions of (1.1). Furthermore, if $u \in H$ is a solution of (1.1) and $u^\pm \neq 0$, then u is a sign-changing solution of (1.1), where

$$u^+(x) = \max\{u(x), 0\} \quad \text{and} \quad u^-(x) = \min\{u(x), 0\}.$$

Our goal in this paper is to seek the least energy sign-changing solutions of (1.1). When $s = 1$, $b = 0$ and $a = 1$, Eq. (1.1) turns out to be (1.5) mentioned above. There are several ways in the literature to obtain sign-changing solution for (1.5) (see [5, 6, 8, 28]). With the application of the minimax arguments in the presence of invariant sets of a descending flow, Bartsch-Liu-Weth [5] obtained a sign-changing solution for (1.5) when f satisfies the classical Ambrosetti-Rabinowitz condition [3] and conditions imposed on $V(x)$ to ensure the compact embedding. However, all these methods heavily rely on the following two decompositions:

$$J(u) = J(u^+) + J(u^-), \quad (1.12)$$

$$\langle J'(u), u^+ \rangle = \langle J'(u^+), u^+ \rangle \quad \text{and} \quad \langle J'(u), u^- \rangle = \langle J'(u^-), u^- \rangle, \quad (1.13)$$

where J is the energy functional of (1.5) given by

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x) u^2) dx - \int_{\mathbb{R}^N} F(x, u) dx. \quad (1.14)$$

When $b > 0$, due to the nonlocal term $(a + b \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 dx)(-\Delta)^s u$, the functional do not possesses the same decompositions as (1.12) and (1.13). Indeed, since $\langle u^+, u^- \rangle_{H^s(\mathbb{R}^N)} > 0$ when $u^\pm \neq 0$, a straightforward computation yields that

$$I_b(u) > I_b(u^+) + I_b(u^-), \quad (1.15)$$

$$\langle I'_b(u), u^+ \rangle > \langle I'_b(u^+), u^+ \rangle \quad \text{and} \quad \langle I'_b(u), u^- \rangle > \langle I'_b(u^-), u^- \rangle. \quad (1.16)$$

Therefore, the methods to obtain sign-changing solutions for the local problem (1.5) seem not be applicable to problem (1.1). Motivated by the work in [7], to get least energy sign-changing solutions, we define the following constrained set:

$$\mathcal{N}_{nod}^b = \{u \in H \mid u^\pm \neq 0, \langle I'_b(u), u^\pm \rangle = 0\}, \quad (1.17)$$

and consider a minimization problem of I_b on \mathcal{N}_{nod}^b . Comparing with the work in [31], the nonlocal term in (1.1), which can be regarded as a combination of the fractional Laplacian operator $(-\Delta)^s$ and the nonlocal term $(\int_{\mathbb{R}^N} |\nabla u|^2 dx) \Delta u$ that appeared in general Kirchhoff type problem (1.2), becomes even more complicated. This will cause some difficulties in proving that \mathcal{N}_{nod}^b is nonempty. Indeed, Shuai [31] proved $\mathcal{N}_{nod}^b \neq \emptyset$ by using the parametric method and the implicit theorem. However, it seems that the method in [31] is not suitable for problem (1.1) due to the complexity of the nonlocal term there. Fortunately, inspired by [2], we prove $\mathcal{N}_{nod}^b \neq \emptyset$ via a modified Miranda's theorem (see Lemma 2.3). Eventually, we prove that the minimizer of the constrained problem is also a sign-changing solution via the quantitative deformation lemma (see [36]) and degree theory (see [7]).

The main results can be stated as follows.

Theorem 1.1. *Let (f_1) – (f_4) hold. Then problem (1.1) possesses one least energy sign-changing solution u_b .*

Another goal of this paper is to prove that the energy of any sign-changing solution of (1.1) is strictly larger than twice the ground state energy, this property is so-called energy doubling by Weth [35]. Consider the semilinear equation (1.5), the conclusion is trivial. Indeed, we denote the Nehari manifold associated to (1.5) by

$$\mathcal{N} = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid \langle J'(u), u \rangle = 0\}$$

and define

$$c := \inf_{u \in \mathcal{N}} J(u). \quad (1.18)$$

Then, for any sign-changing solution $u \in H$ for equation (1.5), it is easy to show that $u^\pm \in \mathcal{N}$. Moreover, if the nonlinearity $f(x, s)$ satisfies some conditions (see [7]) which is analogous to (f_1) – (f_4) , we can deduce that

$$J(w) = J(w^+) + J(w^-) \geq 2c. \quad (1.19)$$

We point out that the minimizer of (1.18) is indeed a ground state solution of the problem (1.5) and $c > 0$ is the least energy of all weak solutions of (1.5). Therefore, by (1.19), it implies that the energy of any sign-changing solution of (1.5) is larger than twice the least energy. When $s = 1$ and $b > 0$, a similar result was obtained by Shuai [31] in a bounded domain Ω . If $0 < s < 1$ and $\Omega = \mathbb{R}^N$, we are interested in that whether the property (1.19) is still true for I_b . To answer this question, we denote the following Nehari manifold associated to (1.1) by

$$\mathcal{N}^b = \{u \in H \setminus \{0\} \mid \langle I_b'(u), u \rangle = 0\},$$

and define

$$c^b := \inf_{u \in \mathcal{N}^b} I_b(u).$$

Then we give the answer via the following theorem:

Theorem 1.2. *Assume $(f_1) - (f_4)$ hold, then $c^b > 0$ is achieved and*

$$I_b(u_b) > 2c^b, \quad (1.20)$$

where u_b is the least energy sign-changing solution obtained in Theorem 1.1. In particular, c^b is achieved either by a positive or a negative function.

It is obvious that the energy of the sign-changing solution u_b obtained in Theorem 1.1 depends on b . It will be of interest to make this dependency precise. This will be the theme of our next result. Namely, we give a convergence property of u_b as $b \rightarrow 0$, which indicates some relationship between $b > 0$ and $b = 0$ for problem (1.1).

Theorem 1.3. *Let f satisfies $(f_1) - (f_4)$. For any sequence $\{b_n\}$ with $b_n \rightarrow 0$ as $n \rightarrow \infty$, there exists a subsequence, still denoted by $\{b_n\}$, such that u_{b_n} converges to u_0 strongly in H as $n \rightarrow \infty$, where u_0 is a least energy sign-changing solution to the following problem*

$$-a(\Delta)^s u + V(x)u = f(x, u) \quad \text{in } \mathbb{R}^N. \quad (1.21)$$

The plan of this paper is as follows: Section 2 reviews some preliminary lemmas, Section 3 covers the proof of the achievement of least energy for the constraint problem (3.20), and Section 4 is devoted to the proofs of our main theorems.

2 Preliminaries

In this section, we first present a modified embedding lemma which is similar to Lemma 2.1 and Theorem 2.1 in [29], and we omit the proof here.

Lemma 2.1. *(i) Suppose that (V_1) holds. Let $q \in [2, 2_s^*]$, then the embeddings*

$$H \hookrightarrow H^s(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$$

are continuous, with $\min\{1, V_0\}\|u\|_{H^s(\mathbb{R}^N)}^2 \leq \|u\|_H^2$ for all $u \in H$. In particular, there exists a constant $C_q > 0$ such that

$$\|u\|_{L^q(\mathbb{R}^N)} \leq C_q \|u\|_H \quad \text{for all } u \in H. \quad (2.1)$$

Moreover, if $q \in [1, 2_s^)$, then the embedding $H \hookrightarrow L^q(B_R)$ is compact for any $R > 0$.*

(ii) Suppose that $(V_1) - (V_2)$ hold. Let $q \in [2, 2_s^)$ be fixed and $\{u_n\}$ be a bounded sequence in H , then there exists $u \in H \cap L^q(\mathbb{R}^N)$ such that, up to a subsequence,*

$$u_n \rightarrow u \quad \text{strongly in } L^q(\mathbb{R}^N) \text{ as } n \rightarrow \infty.$$

Now we recall the Miranda Theorem (see [26]):

Lemma 2.2. *Let $G = \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : |x_i| < L, \text{ for } 1 \leq i \leq N\}$. Suppose that the mapping $F = (f_1, f_2, \dots, f_n) : \overline{G} \rightarrow \mathbb{R}^N$ is continuous on the closure \overline{G} such that $F(x) \neq \theta := (0, 0, \dots, 0)$ for x on the boundary ∂G , and*

$$f_i(x_1, x_2, \dots, x_{i-1}, -L, x_{i+1}, \dots, x_N) \geq 0 \text{ for } 1 \leq i \leq N,$$

$$f_i(x_1, x_2, \dots, x_{i-1}, +L, x_{i+1}, \dots, x_N) \leq 0 \text{ for } 1 \leq i \leq N,$$

then $F(x) = 0$ has a solution in G .

Inspired by the Miranda Theorem above, we have the following variant of Miranda Lemma:

Lemma 2.3. *Let $\mathcal{G} = \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : r < x_i < R, \text{ for } 1 \leq i \leq N\}$. Suppose that the mapping $F = (f_1, f_2, \dots, f_n) : \overline{\mathcal{G}} \rightarrow \mathbb{R}^N$ is continuous on the closure $\overline{\mathcal{G}}$ such that $F(x) \neq \theta := (0, 0, \dots, 0)$ for x on the boundary $\partial \mathcal{G}$, and*

$$f_i(x_1, x_2, \dots, x_{i-1}, r, x_{i+1}, \dots, x_N) \geq 0 \text{ for } 1 \leq i \leq N,$$

$$f_i(x_1, x_2, \dots, x_{i-1}, R, x_{i+1}, \dots, x_N) \leq 0 \text{ for } 1 \leq i \leq N.$$

Then $F(x) = 0$ has a solution in \mathcal{G} .

Proof. Consider the homotopy

$$H : \overline{\mathcal{G}} \times [0, 1] \subseteq \mathbb{R}^{N+1} \rightarrow \mathbb{R}^N$$

by $H(x, t) = (1 - t)F(x) + t(-x + \vec{a})$ for $\vec{a} = (a_1, a_2, \dots, a_N)$ with $r < a_i < R$, $1 \leq i \leq N$. Then $H(x, t) \neq \theta$ for $x \in \partial \mathcal{G}$ and $t \in [0, 1]$. In fact, $H(x, 0) = F(x) \neq \theta$ since $\theta \notin F(\partial \mathcal{G})$, while $H(x, 1) = -x + \vec{a} \neq \theta$ since $r < a_i < R$, $1 \leq i \leq N$. Finally, we have $H(x, t) = \theta$ for some $t \in (0, 1)$ which in turn implies that $F(x) + t(1 - t)^{-1}(-x + a) = \theta$. But this contradicts the conditions of f_i , $i = 1, \dots, N$, given by the lemma. Therefore, by the homotopy invariant theorem of the degree theory, it follows that

$$\deg[F, \mathcal{G}, \theta] = \deg[H(\cdot, 0), \mathcal{G}, \theta] = \deg[H(\cdot, 1), \mathcal{G}, \theta],$$

where $\deg[F, \mathcal{G}, \theta]$ denotes the topological degree of F at θ related to \mathcal{G} . Hence $|\deg[F, \mathcal{G}, \theta]| = 1 \neq 0$ and the result follows by the Kronecker existence theorem. \square

3 Minimizer of constraint minimization problem

We consider a constraint minimization problem on \mathcal{N}_{nod}^b (defined in (1.17)) to seek a critical point of I_b in this section. We begin this section by showing that the set \mathcal{N}_{nod}^b is nonempty. Since the nonlocal term $\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 dx (-\Delta)^s u$ is much more complicated than $\int_{\mathbb{R}^N} |\nabla u|^2 dx \Delta u$, the method in [31] is no longer applicable here. Therefore, we take a different route, namely, we make use of Miranda theorem.

As a start, we define

$$\begin{aligned} A^+(u) &:= \int_{\mathbb{R}^N} |(-\Delta)^{s/2}(u^+)|^2 dx, & A^-(u) &:= \int_{\mathbb{R}^N} |(-\Delta)^{s/2}(u^-)|^2 dx, \\ B^+(u) &:= \int_{\mathbb{R}^N} V(x)(u^+)^2 dx, & B^-(u) &:= \int_{\mathbb{R}^N} V(x)(u^-)^2 dx, \\ C(u) &:= \int_{\mathbb{R}^N} (-\Delta)^{s/2}(u^+)(-\Delta)^{s/2}(u^-) dx. \end{aligned}$$

Lemma 3.1. *Assume that $(f_1) - (f_4)$ hold. Let $u \in H$ with $u^\pm \neq 0$, then there is a unique pair (α_u, β_u) of positive numbers such that $\alpha_u u^+ + \beta_u u^- \in \mathcal{N}_{nod}^b$.*

Proof. Fix an $u \in H$ with $u^\pm \neq 0$. We first establish the existence of α_u and β_u . Consider the vector field

$$W(\alpha, \beta) = (\langle I'_b(\alpha u^+ + \beta u^-), \alpha u^+ \rangle, \langle I'_b(\alpha u^+ + \beta u^-), \beta u^- \rangle)$$

for $\alpha, \beta > 0$, where

$$\begin{aligned} &\langle I'_b(\alpha u^+ + \beta u^-), \alpha u^+ \rangle \\ &= a \int_{\mathbb{R}^N} (-\Delta)^{s/2}(\alpha u^+ + \beta u^-)(-\Delta)^{s/2}(\alpha u^+) dx + \int_{\mathbb{R}^N} V(x)(\alpha u^+)^2 dx \\ &\quad + b \int_{\mathbb{R}^N} |(-\Delta)^{s/2}(\alpha u^+ + \beta u^-)|^2 dx \int_{\mathbb{R}^N} (-\Delta)^{s/2}(\alpha u^+ + \beta u^-)(-\Delta)^{s/2}(\alpha u^+) dx \\ &\quad - \int_{\mathbb{R}^N} f(x, \alpha u^+) \alpha u^+ dx, \end{aligned}$$

and

$$\begin{aligned} &\langle I'_b(\alpha u^+ + \beta u^-), \beta u^- \rangle \\ &= a \int_{\mathbb{R}^N} (-\Delta)^{s/2}(\alpha u^+ + \beta u^-)(-\Delta)^{s/2}(\beta u^-) dx + \int_{\mathbb{R}^N} V(x)(\beta u^-)^2 dx \\ &\quad + b \int_{\mathbb{R}^N} |(-\Delta)^{s/2}(\alpha u^+ + \beta u^-)|^2 dx \int_{\mathbb{R}^N} (-\Delta)^{s/2}(\alpha u^+ + \beta u^-)(-\Delta)^{s/2}(\beta u^-) dx \\ &\quad - \int_{\mathbb{R}^N} f(x, \beta u^-) \beta u^- dx. \end{aligned}$$

By a straightforward computation, we get

$$\begin{aligned}
\langle I'_b(\alpha u^+ + \beta u^-), \alpha u^+ \rangle &= b(\alpha^2 A^+(u) + 2\alpha\beta C(u) + \beta^2 A^-(u))(\alpha^2 A^+(u) + \alpha\beta C(u)) \\
&\quad + a(\alpha^2 A^+(u) + \alpha\beta C(u)) + \alpha^2 B^+(u) - \int_{\mathbb{R}^N} f(x, \alpha u^+) \alpha u^+ dx \\
&= \alpha\beta a C(u) + \alpha\beta^3 b A^-(u) C(u) + \alpha^2 a A^+(u) \\
&\quad + 2\alpha^2 \beta^2 b C^2(u) + \alpha^2 \beta^2 b A^+(u) A^-(u) + \alpha^2 B^+(u) \\
&\quad + 3\alpha^3 \beta b A^+(u) C(u) + \alpha^4 b (A^+(u))^2 - \int_{\mathbb{R}^N} f(x, \alpha u^+) \alpha u^+ dx, \quad (3.1)
\end{aligned}$$

and

$$\begin{aligned}
\langle I'_b(\alpha u^+ + \beta u^-), \beta u^- \rangle &= b(\beta^2 A^-(u) + 2\beta\alpha C(u) + \alpha^2 A^+(u))(\beta^2 A^-(u) + \beta\alpha C(u)) \\
&\quad + a(\beta^2 A^-(u) + \beta\alpha C(u)) + \beta^2 B^-(u) - \int_{\mathbb{R}^N} f(x, \beta u^-) \beta u^- dx \\
&= \alpha\beta a C(u) + \alpha^3 \beta b A^+(u) C(u) + \beta^2 a A^-(u) \\
&\quad + 2\alpha^2 \beta^2 b C^2(u) + \alpha^2 \beta^2 b A^-(u) A^+(u) + \beta^2 B^-(u) \\
&\quad + 3\alpha\beta^3 b A^-(u) C(u) + \beta^4 b (A^-(u))^2 - \int_{\mathbb{R}^N} f(x, \beta u^-) \beta u^- dx. \quad (3.2)
\end{aligned}$$

We will show that there exists $r \in (0, R)$ such that

$$\langle I'_b(ru^+ + \beta u^-), ru^+ \rangle > 0, \quad \langle I'_b(\alpha u^+ + ru^-), ru^- \rangle > 0, \quad \forall \alpha, \beta \in [r, R], \quad (3.3)$$

and

$$\langle I'_b(Ru^+ + \beta u^-), Ru^+ \rangle < 0, \quad \langle I'_b(\alpha u^+ + Ru^-), Ru^- \rangle < 0, \quad \forall \alpha, \beta \in [r, R]. \quad (3.4)$$

Indeed, it follows from the assumption (f_1) that for any $\varepsilon > 0$, there exists a positive constant C_ε such that

$$\int_{\mathbb{R}^N} f(x, \alpha u^+) \alpha u^+ dx \leq \varepsilon \int_{\mathbb{R}^N} |\alpha u^+|^2 dx + C_\varepsilon \int_{\mathbb{R}^N} |\alpha u^+|^{2^*} dx. \quad (3.5)$$

Choosing $\varepsilon = \frac{1}{2}aA^+(u)$, together with (3.1) and (3.5), we deduce that

$$\begin{aligned}
\langle I'_b(\alpha u^+ + \beta u^-), \alpha u^+ \rangle &\geq \alpha\beta a C(u) + \alpha\beta^3 b A^-(u) C(u) + 2\alpha^2 \beta^2 b C^2(u) \\
&\quad + \alpha^2 \beta^2 b A^+(u) A^-(u) + \alpha^2 B^+(u) + \alpha^4 b (A^+(u))^2 \\
&\quad + 3\alpha^3 \beta b A^+(u) C(u) + \frac{\alpha^2 a}{2} (A^+(u))^2 - C_1 \int_{\mathbb{R}^N} |\alpha u^+|^{2^*} dx, \quad (3.6)
\end{aligned}$$

where C_1 is a positive constant. On the other hand, since $u^+ \neq 0$, there exists a constant $\delta > 0$ such that $\text{meas}\{x \in \mathbb{R}^N : u^+(x) > \delta\} > 0$. In addition, by (f_3) and (f_4) , we conclude

that for any $L > 0$, there exists $T > 0$ such that $f(x, s)/s^3 > L$ for all $s > T$. Therefore, for $\alpha > T/\delta$, we have

$$\int_{\mathbb{R}^N} f(x, \alpha u^+) \alpha u^+ dx \geq \int_{\{u^+(x) > \delta\}} \frac{f(x, \alpha u^+)}{(\alpha u^+)^3} (\alpha u^+)^4 \geq L \alpha^4 \int_{\{u^+(x) > \delta\}} (u^+)^4 dx. \quad (3.7)$$

Choose L sufficiently large so that $L \int_{\{u^+(x) > \delta\}} (u^+)^4 dx > 2b(A^+(u))$. By (3.1) and (3.7), we have that

$$\begin{aligned} \langle I'_b(\alpha u^+ + \beta u^-), \alpha u^+ \rangle &\leq \alpha \beta a C(u) + \alpha \beta^3 b A^-(u) C(u) + \alpha^2 a A^+(u) \\ &\quad + 2\alpha^2 \beta^2 b C^2(u) + \alpha^2 \beta^2 b A^+(u) A^-(u) + \alpha^2 B^+(u) \\ &\quad + 3\alpha^3 \beta b A^+(u) C(u) - \alpha^4 b (A^+(u))^2. \end{aligned} \quad (3.8)$$

Similarly, we can obtain

$$\begin{aligned} \langle I'_b(\alpha u^+ + \beta u^-), \beta u^- \rangle &\geq \alpha \beta a C(u) + \alpha^3 \beta b A^+(u) C(u) + 2\alpha^2 \beta^2 b C^2(u) \\ &\quad + \alpha^2 \beta^2 b A^-(u) A^+(u) + \beta^2 B^-(u) + \beta^4 b (A^-(u))^2 \\ &\quad + 3\alpha \beta^3 b A^-(u) C(u) + \frac{\beta^2 b}{2} (A^-(u))^2 - C_2 \int_{\mathbb{R}^N} |\beta u^-|^{2_s^*} dx, \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} \langle I'_b(\alpha u^+ + \beta u^-), \beta u^- \rangle &\leq \alpha \beta a C(u) + \alpha^3 \beta b A^+(u) C(u) + \beta^2 a A^-(u) \\ &\quad + 2\alpha^2 \beta^2 b C^2(u) + \alpha^2 \beta^2 b A^-(u) A^+(u) + \beta^2 B^-(u) \\ &\quad + 3\alpha \beta^3 b A^-(u) C(u) - \beta^4 b (A^-(u))^2, \end{aligned} \quad (3.10)$$

where C_2 is a positive constant. Hence, in view of (3.6) and (3.8)-(3.10), we have that there exists $r \in (0, R)$ such that (3.3) and (3.4) hold. It follows, in view of Lemma 2.3, that there exists $(\alpha_u, \beta_u) \in [r, R] \times [r, R]$ which satisfies $W(\alpha_u, \beta_u) = (0, 0)$, i.e., $\alpha_u u^+ + \beta_u u^- \in \mathcal{N}_{nod}^b$.

It remains to establish the uniqueness of the pair (α_u, β_u) and we need to consider two cases.

Case 1. $u \in \mathcal{N}_{nod}^b$.

If $u \in \mathcal{N}_{nod}^b$, then $u^+ + u^- = u \in \mathcal{N}_{nod}^b$. This in turn implies that

$$a(A^+(u) + C(u)) + b(A^+(u) + 2C(u) + A^-(u))(A^+(u) + C(u)) + B^+(u) = \int_{\mathbb{R}^N} f(x, u^+) u^+ dx, \quad (3.11)$$

and

$$a(A^-(u) + C(u)) + b(A^-(u) + 2C(u) + A^+(u))(A^-(u) + C(u)) + B^-(u) = \int_{\mathbb{R}^N} f(x, u^-) u^- dx. \quad (3.12)$$

We claim that $(\alpha_u, \beta_u) = (1, 1)$ is the unique pair of numbers such that $\alpha_u u^+ + \beta_u u^- \in \mathcal{N}_{nod}^b$.

In fact, let $(\tilde{\alpha}_u, \tilde{\beta}_u)$ be a pair of numbers such that $\tilde{\alpha}_u u^+ + \tilde{\beta}_u u^- \in \mathcal{N}_{nod}^b$. By (3.1) and (3.2), we have

$$\begin{aligned} & b(\tilde{\alpha}_u^2 A^+(u) + 2\tilde{\alpha}_u \tilde{\beta}_u C(u) + \tilde{\beta}_u^2 A^-(u))(\tilde{\alpha}_u^2 A^+(u) + \tilde{\alpha}_u \tilde{\beta}_u C(u)) + \tilde{\alpha}_u^2 B(u) \\ & + a(\tilde{\alpha}_u^2 A^+(u) + \tilde{\alpha}_u \tilde{\beta}_u C(u)) = \int_{\mathbb{R}^N} f(x, \tilde{\alpha}_u u^+) \tilde{\alpha}_u u^+ dx, \end{aligned} \quad (3.13)$$

and

$$\begin{aligned} & b(\tilde{\beta}_u^2 A^-(u) + 2\tilde{\beta}_u \tilde{\alpha}_u C(u) + \tilde{\alpha}_u^2 A^+(u))(\tilde{\beta}_u A^-(u) + \tilde{\beta}_u \tilde{\alpha}_u C(u)) + \tilde{\beta}_u^2 B^-(u) \\ & + a(\tilde{\beta}_u^2 A^-(u) + \tilde{\beta}_u \tilde{\alpha}_u C(u)) = \int_{\mathbb{R}^N} f(x, \tilde{\beta}_u u^-) \tilde{\beta}_u u^- dx. \end{aligned} \quad (3.14)$$

Without loss of generality, we may assume that $0 < \tilde{\alpha}_u \leq \tilde{\beta}_u$. Then (3.13) leads to that

$$\begin{aligned} & b\tilde{\alpha}_u^4 (A^+(u) + 2C(u) + A^-(u))(A^+(u) + C(u)) + \tilde{\alpha}_u^2 (aA^+(u) + aC(u) + B^+(u)) \\ & \leq \int_{\mathbb{R}^N} f(x, \tilde{\alpha}_u u^+) \tilde{\alpha}_u u^+ dx. \end{aligned} \quad (3.15)$$

From (3.11) and (3.15) we see that

$$(\tilde{\alpha}_u - 2 - 1)(aA^+(u) + aC(u) + B^+(u)) \leq \int_{\mathbb{R}^N} \left(\frac{f(x, \tilde{\alpha}_u u^+)}{(\tilde{\alpha}_u u^+)^3} - \frac{f(x, u^+)}{(u^+)^3} \right) (u^+)^4 dx. \quad (3.16)$$

By (f_4) together with (3.16), it implies that $1 \leq \tilde{\alpha}_u \leq \tilde{\beta}_u$. Using the same method, we can get $\tilde{\beta}_u \leq 1$ by (3.12) and (3.14), which implies $\tilde{\alpha}_u = \tilde{\beta}_u = 1$.

Case 2. $u \notin \mathcal{N}_{nod}^b$.

If $u \notin \mathcal{N}_{nod}^b$, there exists a pair (α_u, β_u) of positive numbers such that $\alpha_u u^+ + \beta_u u^- \in \mathcal{N}_{nod}^b$. Suppose that there exists another pair (α'_u, β'_u) of positive numbers such that $\alpha'_u u^+ + \beta'_u u^- \in \mathcal{N}_{nod}^b$. Set $v := \alpha_u u^+ + \beta_u u^-$ and $v' := \alpha'_u u^+ + \beta'_u u^-$, we have

$$\frac{\alpha'_u}{\alpha_u} v^+ + \frac{\beta'_u}{\beta_u} v^- = \alpha'_u u^+ + \beta'_u u^- = v' \in \mathcal{N}_{nod}^b.$$

Noticing that $v \in \mathcal{N}_{nod}^b$, we obtain that $\alpha'_u = \alpha_u$ and $\beta'_u = \beta_u$, which implies that (α_u, β_u) is the unique pair of numbers such that $\alpha_u u^+ + \beta_u u^- \in \mathcal{N}_{nod}^b$. Hence we finish our proof. \square

Lemma 3.2. Assume $(f_1) - (f_4)$ hold, and $u \in H$ such that $\langle I'_b(u), u^+ \rangle \leq 0$ and $\langle I'_b(u), u^- \rangle \leq 0$. Then the unique pair (α_u, β_u) obtained in Lemma 3.1 satisfies $0 < \alpha_u, \beta_u \leq 1$.

Proof. Without loss of generality, we may assume that $\alpha_u \geq \beta_u > 0$. Since $\alpha_u u^+ + \beta_u u^- \in \mathcal{N}_{nod}^b$, we have

$$\begin{aligned}
& b\alpha_u^4(A^+(u) + 2C(u) + A^-(u))(A^+(u) + C(u)) + \alpha_u^2(aA^+(u) + aC(u) + B^+(u)) \\
& \geq b(\alpha_u^2 A^+(u) + 2\alpha_u \beta_u C(u) + \beta_u^2 A^-(u))(\alpha_u^2 A^+(u) + \alpha_u \beta_u C(u)) \\
& \quad + (a\alpha_u^2 A^+(u) + a\alpha_u \beta_u C(u) + \alpha_u B^+(u)) \\
& = \int_{\mathbb{R}^N} f(x, \alpha_u u^+) \alpha_u u^+ dx. \tag{3.17}
\end{aligned}$$

Sine $\langle I'_b(u), u^- \rangle \leq 0$, it yields that

$$b(A^+(u) + 2C(u) + A^-(u))(A^+(u) + C(u)) + a(A^+(u) + C(u) + B^+(u)) \leq \int_{\mathbb{R}^N} f(u^+) u^+ dx. \tag{3.18}$$

Then, it follows from (3.17) and (3.18) that

$$(\alpha_u^{-2} - 1)(aA^+(u) + aC(u) + B^+(u)) \leq \int_{\mathbb{R}^N} \left(\frac{f(\alpha_u u^+)}{(\alpha_u u^+)^3} - \frac{f(u^+)}{(u^+)^3} \right) (u^+)^4 dx. \tag{3.19}$$

If $\alpha_u > 1$, the left side of (3.19) is negative while, from (f_4) , the right side of (3.19) is positive. This implies that $\alpha_u \leq 1$, which completes the proof. \square

Lemma 3.3. *For any fixed $u \in H$ with $u^\pm \neq 0$, (α_u, β_u) is the unique maximum point of the function $\phi : (\mathbb{R}_+ \times \mathbb{R}_+) \rightarrow \mathbb{R}$, where (α_u, β_u) is obtained in Lemma 3.1, and $\phi(\alpha, \beta) := I_b(\alpha u^+ + \beta u^-)$.*

Proof. From the proof of Lemma 3.1, (α_u, β_u) is the unique critical point of ϕ in $(\mathbb{R}_+ \times \mathbb{R}_+)$. By (f_3) , we conclude that $\phi(\alpha, \beta) \rightarrow -\infty$ uniformly as $|(\alpha, \beta)| \rightarrow \infty$. Therefore, it is sufficient to check that the maximum point cannot be achieved on the boundary of $(\mathbb{R}_+ \times \mathbb{R}_+)$. We use the argument of contradiction. Suppose that $(0, \bar{\beta})$ is the global maximum point of ϕ with $\bar{\beta} \geq 0$. By a direct calculation, we have

$$\begin{aligned}
\phi(\alpha, \bar{\beta}) &= I_b(\alpha u^+ + \bar{\beta} u^-) \\
&= \frac{a}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2}(\alpha u^+ + \bar{\beta} u^-)|^2 dx + \int_{\mathbb{R}^N} V(x)(\alpha u^+ + \bar{\beta} u^-)^2 dx \\
&\quad + \frac{b}{4} \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}(\alpha u^+ + \bar{\beta} u^-)|^2 dx \right)^2 - \int_{\mathbb{R}^N} F(x, \alpha u^+ + \bar{\beta} u^-) dx,
\end{aligned}$$

which implies that ϕ is an increasing function with respect to α if α is small enough. This yields the contradiction. Similarly, ϕ can not achieve its global maximum on $(\alpha, 0)$ for any $\alpha \geq 0$. We finish the proof. \square

Next we consider the following minimization problem

$$c_{nod}^b := \inf \{I_b(u) : u \in \mathcal{N}_{nod}^b\}. \quad (3.20)$$

As \mathcal{N}_{nod}^b is nonempty in H by Lemma 2.1, we see that c_{nod}^b is well defined.

Lemma 3.4. *Assume that $(f_1) - (f_4)$ hold, then c_{nod}^b can be achieved.*

Proof. For every $u \in \mathcal{N}_{nod}^b$, we have $\langle I'_b(u), u \rangle = 0$. Then by $(f_1), (f_2)$ and Lemma 2.1, we get

$$\begin{aligned} a\|u\|_H^2 &\leq a \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 dx + \int_{\mathbb{R}^N} V(x)|u|^2 dx + b \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 dx \right)^2 \\ &= \int_{\mathbb{R}^N} f(x, u) u dx \\ &\leq \varepsilon \int_{\mathbb{R}^N} |u|^2 dx + C_\varepsilon \int_{\mathbb{R}^N} |u|^p dx, \end{aligned} \quad (3.21)$$

where C_ε is a positive constant depending only on ε . Choosing ε sufficiently small such that $\varepsilon\|u\|_{L^2(\mathbb{R}^N)}^2 \leq \frac{a}{2}\|u\|_H^2$, we can then deduce that there exists a constant $\mu > 0$ such that $\|u\|_H^2 \geq \mu$. And by (f_4) , for $s \neq 0$, we have

$$f(x, s) - 4F(x, s) \geq 0.$$

Hence

$$I_b(u) = I_b(u) - \frac{1}{4} \langle I'_b(u), u \rangle \geq \frac{a}{4} \|u\|_H^2 \geq \frac{a\mu}{4}, \quad (3.22)$$

which implies that $c_{nod}^b \geq \frac{1}{4}\mu > 0$.

Let $\{u_n\} \subset \mathcal{N}_{nod}^b$ be a minimizing sequence such that $I_b(u_n) \rightarrow c_{nod}^b$. By (3.22), we have $\{u_n\}$ is bounded in H . Utilizing Lemma 2.1 and the properties of L^p -space, up to a subsequence, we have

$$\begin{aligned} u_n^\pm &\rightharpoonup u_b^\pm \text{ weakly in } H, & u_n^\pm &\rightarrow u_b^\pm \text{ a.e. in } \mathbb{R}^N, \\ u_n^\pm &\rightarrow u_b^\pm \text{ strongly in } L^q(\mathbb{R}^N), & \text{for } q &\in [2, 2_s^*), \\ (-\Delta)^{s/2} u_n^\pm &\rightarrow (-\Delta)^{s/2} u_b^\pm \text{ a.e. in } \mathbb{R}^N. \end{aligned} \quad (3.23)$$

Furthermore, the conditions $(f_1) - (f_2)$ combined with the compactness lemma of Strauss [32] gives that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(x, u_n^\pm) u_n^\pm dx = \int_{\mathbb{R}^N} f(x, u_b^\pm) u_b^\pm dx, \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} F(x, u_n^\pm) dx = \int_{\mathbb{R}^N} F(x, u_b^\pm) dx. \quad (3.24)$$

Since $u_n \in \mathcal{N}_{nod}^b$, we have $\langle I'_b(u_n), u_n^\pm \rangle = 0$, that is,

$$\begin{aligned} & a \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}(u_n^+)|^2 dx + \int_{\mathbb{R}^N} (-\Delta)^{s/2}(u_n^+) (-\Delta)^{s/2}(u_n^-) dx \right) + \int_{\mathbb{R}^N} V(x) |u_n^+|^2 dx \\ & + b \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_n|^2 dx \right) \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}(u_n^+)|^2 dx + \int_{\mathbb{R}^N} (-\Delta)^{s/2}(u_n^+) (-\Delta)^{s/2}(u_n^-) dx \right) \\ & = \int_{\mathbb{R}^N} f(x, u_n^+) u_n^+ dx, \end{aligned} \quad (3.25)$$

and

$$\begin{aligned} & a \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}(u_n^-)|^2 dx + \int_{\mathbb{R}^N} (-\Delta)^{s/2}(u_n^-) (-\Delta)^{s/2}(u_n^+) dx \right) + \int_{\mathbb{R}^N} V(x) |u_n^+|^2 dx \\ & + b \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_n|^2 dx \right) \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}(u_n^-)|^2 dx + \int_{\mathbb{R}^N} (-\Delta)^{s/2}(u_n^-) (-\Delta)^{s/2}(u_n^+) dx \right) \\ & = \int_{\mathbb{R}^N} f(u_n^+) u_n^+ dx. \end{aligned} \quad (3.26)$$

With a similar argument as (3.21), there exists a constant $l > 0$ such that $\|u_n^\pm\|_H^2 \geq l$ for all $n \in \mathbb{N}$. From (3.25), we obtain that $\int_{\mathbb{R}^N} f(x, u_n^\pm) u_n^\pm \geq l$. Hence by (3.24), we have $\int_{\mathbb{R}^N} f(u_b^\pm) u_b^\pm \geq l$, i.e. $u_b^\pm \neq 0$.

From Lemma 3.1, there exist $\alpha_{u_b}, \beta_{u_b} > 0$ such that $\bar{u}_b := \alpha_{u_b} u_b^+ + \beta_{u_b} u_b^- \in \mathcal{N}_{nod}^b$, that is,

$$\langle I'_b(\alpha_{u_b} u_b^+ + \beta_{u_b} u_b^-), \alpha_{u_b} u_b^+ \rangle = \langle I'_b(\alpha_{u_b} u_b^+ + \beta_{u_b} u_b^-), \beta_{u_b} u_b^- \rangle = 0.$$

Now, we show that $\alpha_{u_b}, \beta_{u_b} \leq 1$. In fact, by (3.23)-(3.25) and Fatou's Lemma, we conclude that

$$\begin{aligned} & a \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}(u^+)|^2 dx + \int_{\mathbb{R}^N} (-\Delta)^{s/2}(u^+) (-\Delta)^{s/2}(u^-) dx \right) + \int_{\mathbb{R}^N} V(x) |u^+|^2 dx \\ & + b \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 dx \right) \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}(u^+)|^2 dx + \int_{\mathbb{R}^N} (-\Delta)^{s/2}(u^+) (-\Delta)^{s/2}(u^-) dx \right) \\ & \leq \int_{\mathbb{R}^N} f(x, u^+) u^+ dx. \end{aligned} \quad (3.27)$$

It follows from Lemma 2.2 and (3.27) that $\alpha_{u_b} \leq 1$. Similarly, $\beta_{u_b} \leq 1$. It follows from (f_4) that $H(s) := sf(x, s) - 4F(x, s)$ is a non-negative function, which is also increasing in $|s|$. Hence we have

$$\begin{aligned} c_{nod}^b & \leq I(\bar{u}_b) = I(\bar{u}_b) - \frac{1}{4} \langle I'(\bar{u}_b), \bar{u}_b \rangle \\ & = \frac{a}{4} \|\bar{u}_b\|_H^2 + \frac{1}{4} \int_{\mathbb{R}^N} \left(f(\bar{u}_b) \bar{u}_b - 4F(\bar{u}_b) \right) dx \\ & = \frac{a\alpha_{u_b}^2}{4} \|u_b^+\|_H^2 + \frac{a\alpha_{u_b}\beta_{u_b}}{2} \int_{\mathbb{R}^N} (-\Delta)^{s/2}(u_b^+) (-\Delta)^{s/2}(u_b^-) dx + \frac{a\beta_{u_b}^2}{4} \|u_b^-\|_H^2 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \int_{\mathbb{R}^N} \left(f(\alpha_{u_b} u_b^+) \alpha_{u_b} u_b^+ - 4F(\alpha_{u_b} u_b^+) \right) dx + \frac{1}{4} \int_{\mathbb{R}^N} \left(f(\beta_{u_b} u_b^-) \beta_{u_b} u_b^- - 4F(\beta_{u_b} u_b^-) \right) dx \\
& \leq \frac{a}{4} \|u_b\|_H^2 + \frac{1}{4} \int_{\mathbb{R}^N} \left(f(u_b) u_b - 4F(u_b) \right) dx \\
& \leq \liminf_{n \rightarrow \infty} \left[I_b(u_n) - \frac{1}{4} \langle I'_b(u_n), u_n \rangle \right] = c_{nod}^b.
\end{aligned}$$

From the analysis above, we conclude that $\alpha_{u_b} = \beta_{u_b} = 1$. Thus, $\bar{u}_b = u_b$ and $I_b(u_b) = c_{nod}^b$. \square

4 Proofs of main theorems

This section is devoted to prove our main results. We first prove that the minimizer u_b for the problem (3.20) is indeed a sign-changing solution of (1.1), that is, $I'_b(u_b) = 0$.

Proof of Theorem 1.1. Applying the quantitative deformation lemma (see [36]), we want to prove that $I'_b(u_b) = 0$. We first recall that $\langle I'_b(u_b), u_b^+ \rangle = \langle I'_b(u_b), u_b^- \rangle = 0$. Then it follows from Lemma 3.3 that

$$I_b(\alpha u_b^+ + \beta u_b^-) < I_b(u_b^+ + u_b^-) = c_{nod}^b \quad (4.1)$$

for $(\alpha, \beta) \in (\mathbb{R}_+ \times \mathbb{R}_+)$ and $(\alpha, \beta) \neq (1, 1)$. If $I'_b(u_b) \neq 0$, then there exist $\delta > 0$ and $\lambda > 0$ such that

$$\|I'_b(u_b)\| \geq \lambda \quad \text{for all } \|v - u_b\|_H \leq 3\delta.$$

Let $D := (\frac{1}{2}, \frac{3}{2}) \times (\frac{1}{2}, \frac{3}{2})$ and $g(\alpha, \beta) := \alpha u_b^+ + \beta u_b^-$. In view of lemma 3.3 again, we have

$$m^b := \max_{\partial D} I_b \circ g < c_{nod}^b. \quad (4.2)$$

Let $\varepsilon = \min\{(c_{nod}^b - m^b)/2, \lambda\delta/8\}$ and $S = B(u_b, \delta)$. It follows from lemma 2.2 in [36] that there is a continuous mapping η such that

- (a) $\eta(1, u) = u$ if $u \neq I_b^{-1}([c_{nod}^b - 2\varepsilon, c_{nod}^b + 2\varepsilon]) \cap S_{2\delta}$;
- (b) $\eta(1, I_b^{c_{nod}^b + \varepsilon} \cap S) \subset I_b^{c_{nod}^b - \varepsilon}$;
- (c) $I_b(\eta(1, u)) \leq I_b(u)$ for all $u \in H$.

It is clear that

$$\max_{(\alpha, \beta) \in \bar{D}} I_b(\eta(1, g(\alpha, \beta))) < c_{nod}^b. \quad (4.3)$$

Next, we prove that $\eta(1, g(D)) \cap \mathcal{N}_{nod}^b \neq \emptyset$, which contradicts to the definition of c_{nod}^b . Let us define $h(\alpha, \beta) := \eta(1, g(\alpha, \beta))$ and

$$\begin{aligned}
\Psi_0(\alpha, \beta) &:= \left(\langle I'_b(g(\alpha, \beta)), u_b^+ \rangle, \langle I'_b(g(\alpha, \beta)), u_b^- \rangle \right) \\
&= \left(\langle I'_b(\alpha u_b^+ + \beta u_b^-), u_b^+ \rangle, \langle I'_b(\alpha u_b^+ + \beta u_b^-), u_b^- \rangle \right), \\
\Psi_1(\alpha, \beta) &:= \left(\frac{1}{\alpha} \langle I'_b(h(\alpha, \beta)), h^+(\alpha, \beta) \rangle, \frac{1}{\beta} \langle I'_b(h(\alpha, \beta)), h^-(\alpha, \beta) \rangle \right).
\end{aligned}$$

By Lemma 3.1 and the degree theory, this implies that $\deg(\Psi_0, D, 0) = 1$. It follows, in view of (4.2), that $g = h$ on ∂D , from which we obtain $\deg(\Psi_1, D, 0) = \deg(\Psi_0, D, 0) = 1$. Therefore, $\Psi_1(\alpha_0, \beta_0) = 0$ for some $(\alpha_0, \beta_0) \in D$, so that $\eta(1, g(\alpha_0, \beta_0)) = h(\alpha_0, \beta_0) \in \mathcal{N}_{nod}^b$, which is a contradiction. We have thus proved that u_b is a critical point of I_b . Moreover, u_b is a sign-changing solution for problem (1.1). \square

By Theorem 1.1, we obtain a least energy sign-changing solution of problem (1.1). Hence, Theorem 1.2 follows immediately if we establish the strict inequality $c_{nod}^b > 2c^b$, where $c^b = \inf_{u \in \mathcal{N}^b} I(u)$.

Proof of Theorem 1.2. Recall that \mathcal{N}^b denotes the Nehari manifold associated to (1.1) and $c^b = \inf_{u \in \mathcal{N}^b} I(u)$. Then, by a similar argument to that in the proof of Lemma 3.4, there exists $v_b \in \mathcal{N}^b$ such that $I_b(v_b) = c^b > 0$. We assert that v_b is actually a ground state solution of (1.1). In fact, by virtue of section 4 in [36], v_b is a critical point of I_b on H , or equivalently, $I'(v_b) = 0$.

Reviewing Theorem 1.1 again, we know that u_b is a least sign-changing solution of problem (1.1). Utilizing the same method in Lemma 3.1, we conclude that there exists $\alpha_{u_b^+} > 0$ such that $\alpha_{u_b^+} u_b^+ \in \mathcal{N}^b$. And we can prove that there exists $\beta_{u_b^-} > 0$ such that $\beta_{u_b^-} u_b^- \in \mathcal{N}^b$ analogously. Moreover, Lemma 3.3 implies that $\alpha_{u_b^+}, \beta_{u_b^-} \in (0, 1)$.

Therefore, in view of Lemma 3.4, we obtain that

$$2c^b \leq I_b(\alpha_{u_b^+} u_b^+) + I_b(\beta_{u_b^-} u_b^-) \leq I_b(\alpha_{u_b^+} u_b^+ + \beta_{u_b^-} u_b^-) \leq I(u_b^+ + u_b^-) = c_{nod}^b. \quad (4.4)$$

It follows that $c^b > 0$ which cannot be achieved by a sign-changing solution. Hence, we complete the proof. \square

Finally, we close this section with the proof of Theorem 1.3. In the following, we regard $b > 0$ as a parameter in problem (1.1).

Proof of Theorem 1.3. We shall proceed through several claims on analyzing the convergence property of u_b as $b \rightarrow 0$, where u_b is the least energy sign-changing solution obtained in Theorem 1.1.

Claim 1. For any sequence $\{b_n\}$ as $b_n \searrow 0$, $\{u_{b_n}\}$ is bounded in H .

Proof. We select a nonzero function $\psi \in \mathcal{C}_c^\infty$ with $\psi^\pm \neq 0$. By the assumptions (f_3) and (f_4) , we deduce that, for any $b \in [0, 1]$, there exists a pair (λ_1, λ_2) independent of b , such that

$$\langle I'_b(\lambda_1\psi^+ + \lambda_2\psi^-), \lambda_1\psi^+ \rangle < 0 \quad \text{and} \quad \langle I'_b(\lambda_1\psi^+ + \lambda_2\psi^-), \lambda_2\psi^- \rangle < 0.$$

Then by virtue of Lemma 2.2, we get that, for any $b \in [0, 1]$, there exists a unique pair $(\alpha_\psi(b), \beta_\psi(b)) \in (0, 1] \times (0, 1]$ such that

$$\bar{\psi} := \alpha_\psi(b)\lambda_1\psi^+ + \beta_\psi(b)\lambda_2\psi^- \in \mathcal{N}_{nod}^b. \quad (4.5)$$

Recall that (f_1) implies that $f(s) \leq C_1|s| + C_2|s|^{p-1}$ and $F(s) \leq C_1|s|^2 + C_2|s|^p$, where C_1 and C_2 are positive constants. Then it follows that

$$\begin{aligned} I_b(u_b) &\leq I_b(\bar{\psi}) = I_b(\bar{\psi}) - \langle I'_b(\bar{\psi}), \bar{\psi} \rangle \\ &= \frac{a}{4} \|\bar{\psi}\|_H^2 + \int_{\mathbb{R}^N} (f(x, \bar{\psi})\bar{\psi} - 4F(x, \bar{\psi})) dx \\ &= \frac{a}{4} \|\lambda_1\psi^+\|_H^2 + \frac{a}{2} \int_{\mathbb{R}^N} (-\Delta)^{s/2}(\lambda_1\psi^+)(-\Delta)^{s/2}(\lambda_2\psi^-) dx + \frac{a}{4} \|\lambda_2\psi^-\|_H^2 \\ &\quad + C_1 \int_{\mathbb{R}^N} (\lambda_1^2|\psi^+|^2 + \lambda_2^2|\psi^-|^2) dx + C_2 \int_{\mathbb{R}^N} (\lambda_1^p|\psi^+|^p + \lambda_2^p|\psi^-|^p) dx \\ &:= C_0, \end{aligned} \quad (4.6)$$

where C_0 is a positive constant independent of b . Then we write, as $n \rightarrow \infty$,

$$C_0 + 1 \geq I_{b_n}(u_{b_n}) = I_{b_n}(u_{b_n}) - \frac{1}{4} \langle I'_{b_n}(u_{b_n}), u_{b_n} \rangle \geq \frac{a}{4} \|u_{b_n}\|_H^2, \quad (4.7)$$

from which the claim follows. \square

Claim 2. (1.21) possesses one sign-changing solution u_0 .

Proof. Going if necessary to a subsequence, thanks to Lemma 2.1, we conclude that there exists $u_0 \in H$, such that

$$\begin{aligned} u_{b_n} &\rightharpoonup u_0 \quad \text{weakly in } H, \\ u_{b_n} &\rightarrow u_0 \quad \text{strongly in } L^q(\mathbb{R}^N) \quad \text{for } q \in [2, 2_s^*), \\ u_{b_n} &\rightarrow u_0 \quad \text{a.e. in } \mathbb{R}^N. \end{aligned} \quad (4.8)$$

Since u_{b_n} is a weak solution of (1.1) with $b = b_n$, we then have

$$\begin{aligned} &a \int_{\mathbb{R}^N} (-\Delta)^{s/2} u_{b_n} (-\Delta)^{s/2} \phi dx \\ &\quad + b_n \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_{b_n}|^2 dx \int_{\mathbb{R}^N} (-\Delta)^{s/2} u_{b_n} (-\Delta)^{s/2} \phi dx + \int_{\mathbb{R}^N} V(x) u_{b_n}^2 dx \\ &\quad = \int_{\mathbb{R}^N} f(x, u_{b_n}) \phi dx, \end{aligned} \quad (4.9)$$

for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$. From (3.24), (4.8), (4.9) and Claim 1, we see that

$$a \int_{\mathbb{R}^N} (-\Delta)^{s/2} u_0 (-\Delta)^{s/2} \phi dx + \int_{\mathbb{R}^N} V(x) u_0^2 dx = \int_{\mathbb{R}^N} f(x, u_0) \phi dx \quad (4.10)$$

for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$, which yields that u_0 is a weak solution of (1.21). It remains now to establish that $u_0^\pm \neq 0$. Indeed, via a similar argument to that in the proof in Lemma 3.4, we conclude that $\int_{\mathbb{R}^N} f(u_0^\pm) u_0^\pm > c > 0$, where c is a positive constant. Therefore, we complete the proof of the claim. \square

Claim 3. (1.21) possesses a least energy sign-changing solution v_0 , and there exists a unique pair $(\alpha_{b_n}, \beta_{b_n}) \in \mathbb{R}_+ \times \mathbb{R}_+$ such that $\alpha_{b_n} v_0^+ + \beta_{b_n} v_0^- \in \mathcal{N}_{nod}^{b_n}$. Moreover, $(\alpha_{b_n}, \beta_{b_n}) \rightarrow (1, 1)$ as $n \rightarrow \infty$.

Proof. With a similar argument to the proof of Theorem 1.1, we have that (1.21) possesses a least energy sign-changing solution v_0 , where $I(v_0) = c_{nod}^0$ and $I'(v_0) = 0$. Then, by Lemma 2.1, we can easily obtain the existence and uniqueness of the pair $(\alpha_{b_n}, \beta_{b_n})$ such that $\alpha_{b_n} v_0^+ + \beta_{b_n} v_0^- \in \mathcal{N}_{nod}^{b_n}$. Besides, we have $\alpha_{b_n}, \beta_{b_n} > 0$. Then the claim will follow once we have proved that $(\alpha_{b_n}, \beta_{b_n}) \rightarrow (1, 1)$ as $n \rightarrow \infty$. In fact, since $\alpha_{b_n} v_0^+ + \beta_{b_n} v_0^- \in \mathcal{N}_{nod}^{b_n}$, we then have

$$\begin{aligned} & a \left(\alpha_{b_n}^2 \int_{\mathbb{R}^N} |(-\Delta)^{s/2} (v_0^+)|^2 dx + \alpha_{b_n} \beta_{b_n} \int_{\mathbb{R}^N} (-\Delta)^{s/2} (v_0^+) (-\Delta)^{s/2} (v_0^-) dx \right) \\ & + b_n \left(\alpha_{b_n}^2 \int_{\mathbb{R}^N} |(-\Delta)^{s/2} (v_0^-)|^2 dx + \alpha_{b_n} \beta_{b_n} \int_{\mathbb{R}^N} (-\Delta)^{s/2} (v_0^+) (-\Delta)^{s/2} (v_0^-) dx \right) \\ & \times \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2} (\alpha_{b_n} v_0^+ + \beta_{b_n} v_0^-)|^2 dx \right) + \alpha_{b_n}^2 \int_{\mathbb{R}^N} V(x) |v_0^+|^2 dx \\ & = \int_{\mathbb{R}^N} f(x, \alpha_{b_n} v_0^+) \alpha_{b_n} v_0^+ dx, \end{aligned} \quad (4.11)$$

and

$$\begin{aligned} & a \left(\beta_{b_n}^2 \int_{\mathbb{R}^N} |(-\Delta)^{s/2} (v_0^-)|^2 dx + \alpha_{b_n} \beta_{b_n} \int_{\mathbb{R}^N} (-\Delta)^{s/2} (v_0^+) (-\Delta)^{s/2} (v_0^-) dx \right) \\ & + b_n \left(\beta_{b_n}^2 \int_{\mathbb{R}^N} |(-\Delta)^{s/2} (v_0^+)|^2 dx + \alpha_{b_n} \beta_{b_n} \int_{\mathbb{R}^N} (-\Delta)^{s/2} (v_0^+) (-\Delta)^{s/2} (v_0^-) dx \right) \\ & \times \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2} (\alpha_{b_n} v_0^+ + \beta_{b_n} v_0^-)|^2 dx \right) + \alpha_{b_n}^2 \int_{\mathbb{R}^N} V(x) |v_0^+|^2 dx \\ & = \int_{\mathbb{R}^N} f(x, \beta_{b_n} v_0^-) \beta_{b_n} v_0^- dx. \end{aligned} \quad (4.12)$$

From (f₃)-(f₄), and recalling that $b_n \rightarrow 0$ as $n \rightarrow \infty$, we deduce that $\{\alpha_{b_n}\}$ and $\{\beta_{b_n}\}$ are bounded sequences. Up to a subsequence, suppose that $\alpha_{b_n} \rightarrow \alpha_0$ and $\beta_{b_n} \rightarrow \beta_0$, then it

follows from (4.11) and (4.12) that

$$\begin{aligned} a\left(\alpha_0^2 \int_{\mathbb{R}^N} |(-\Delta)^{s/2}(v_0^+)|^2 dx + \alpha_0 \beta_0 \int_{\mathbb{R}^N} (-\Delta)^{s/2}(v_0^+)(-\Delta)^{s/2}(v_0^-) dx\right) \\ + \alpha_0^2 \int_{\mathbb{R}^N} V(x)|v_0^+|^2 dx = \int_{\mathbb{R}^N} f(x, \alpha_0 v_0^+) \alpha_0 v_0^+ dx, \end{aligned} \quad (4.13)$$

and

$$\begin{aligned} a\left(\beta_0^2 \int_{\mathbb{R}^N} |(-\Delta)^{s/2}(v_0^-)|^2 dx + \alpha_0 \beta_0 \int_{\mathbb{R}^N} (-\Delta)^{s/2}(v_0^+)(-\Delta)^{s/2}(v_0^-) dx\right) \\ + \beta_0^2 \int_{\mathbb{R}^N} V(x)|v_0^-|^2 dx = \int_{\mathbb{R}^N} f(x, \beta_0 v_0^-) \beta_0 v_0^- dx. \end{aligned} \quad (4.14)$$

Noticing that v_0 is a solution of (1.21), we then have

$$\begin{aligned} a\left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}(v_0^+)|^2 dx + \int_{\mathbb{R}^N} (-\Delta)^{s/2}(v_0^+)(-\Delta)^{s/2}(v_0^-) dx\right) \\ + \int_{\mathbb{R}^N} V(x)|v_0^+|^2 dx = \int_{\mathbb{R}^N} f(x, v_0^+) v_0^+ dx, \end{aligned} \quad (4.15)$$

and

$$\begin{aligned} a\left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}(v_0^-)|^2 dx + \int_{\mathbb{R}^N} (-\Delta)^{s/2}(v_0^+)(-\Delta)^{s/2}(v_0^-) dx\right) \\ + \int_{\mathbb{R}^N} V(x)|v_0^-|^2 dx = \int_{\mathbb{R}^N} f(x, v_0^-) v_0^- dx. \end{aligned} \quad (4.16)$$

Moreover, by the assumptions (f_3) and (f_4) , we conclude that $f(s)/|s|^3$ is nondecreasing in $|s|$. Thus, in view of (4.13)-(4.16), we can easily check that $(\alpha_0, \beta_0) = (1, 1)$, and the claim follows. \square

We now come back the proof of Theorem 1.3. We assert that u_0 obtained in Claim 2 is a least energy solution of problem (1.2). In fact, by virtue of Claim 3 and Lemma 2.3, we have

$$I_0(v_0) \leq I_0(u_0) = \lim_{n \rightarrow \infty} I_{b_n}(u_{b_n}) \leq \lim_{n \rightarrow \infty} I_{b_n}(\alpha_{b_n} v_0^+ + \beta_{b_n} v_0^-) = \lim_{n \rightarrow \infty} I_0(v_0^+ + v_0^-) = I_0(v_0),$$

which yields Theorem 1.3. \square

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